Correction

Exercice 1 (10 points)

Partie A

1. Calculons I_n :

$$I_n = \frac{1}{n}\left[\sin nx\right]_{\frac{\pi}{2}}^{\pi} = \frac{1}{n}\left[\sin n\pi - \sin\left(n\frac{\pi}{2}\right)\right] = -\frac{1}{n}\sin\left(n\frac{\pi}{2}\right)$$

2. On intègre par parties, en posant :

$$u(x) = x \quad \text{et} \quad v'(x) = \cos nx$$

Donc :

$$J_n = \frac{1}{n}\left[x\sin nx\right]_0^{\frac{\pi}{2}} - \frac{1}{n}\int_0^{\frac{\pi}{2}} \sin nx\mathrm{d}x$$

$$J_n = \frac{1}{n}\frac{\pi}{2}\sin n\frac{\pi}{2} + \frac{1}{n^2}\left[\cos nx\right]_0^{\frac{\pi}{2}}$$

$$J_n = \frac{1}{n}\frac{\pi}{2}\sin n\frac{\pi}{2} + \frac{1}{n^2}\left[\cos n\frac{\pi}{2} - 1\right]$$

$$J_n = \frac{\pi}{2n}\sin n\frac{\pi}{2} + \frac{1}{n^2}\cos n\frac{\pi}{2} - \frac{1}{n^2}$$

3. On obtient :

$$I_1 = -\sin\frac{\pi}{2} = -1 \quad I_2 = -\frac{1}{2}\sin\pi = 0 \quad I_3 = -\frac{1}{3}\sin 3\frac{\pi}{2} = \frac{1}{3}$$

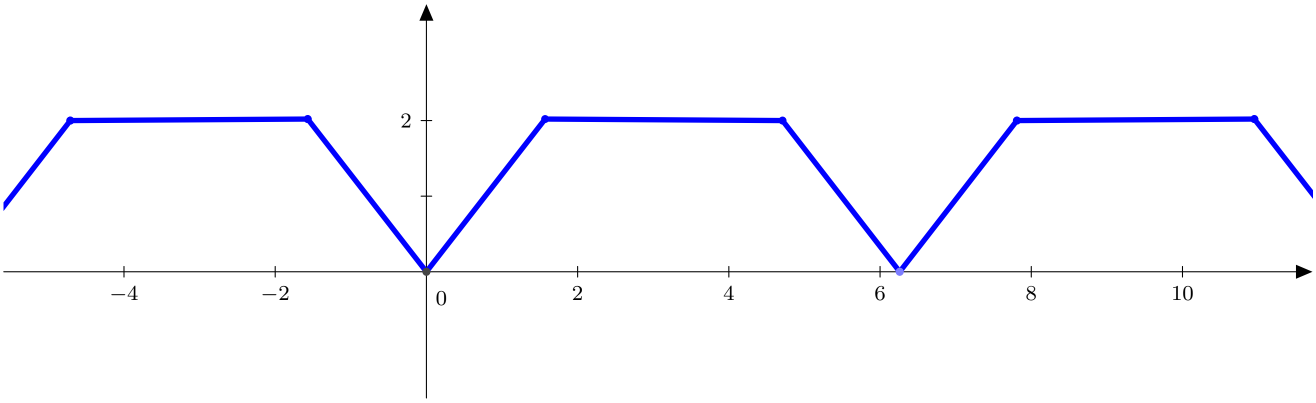
$$J_1 = \frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - 1 = \frac{\pi}{2} - 1$$

$$J2 = \frac{\pi}{4}\sin\pi + \frac{1}{4}\cos\pi - \frac{1}{4} = -\frac{1}{2}$$

$$J_3 = \frac{\pi}{6}\sin 3\frac{\pi}{2} + \frac{1}{9}\cos 3\frac{\pi}{2} - \frac{1}{9} = -\frac{\pi}{6} - \frac{1}{9}$$

Partie B

1. Représentation graphique de la fonction f :



2. (a) Calculons a_0 :

$$a_0 = \frac{1}{T}\int_a^{a+T} f(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} f(t)\mathrm{d}t$$

car f est paire et 2π-périodique ; donc :

$$a_0 = \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{2E}{\pi} t\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi} E\mathrm{d}t\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{2E}{\pi}\left[\frac{t^2}{2}\right]_0^{\frac{\pi}{2}} + E\left[t\right]_{\frac{\pi}{2}}^{\pi}\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{E}{\pi}\frac{\pi^2}{4} + E\frac{\pi}{2}\right] = \frac{1}{\pi}\left[\frac{E\pi}{4} + \frac{E\pi}{2}\right] = \frac{3E}{4}$$

(b) La fonction f est paire donc pour tout entier $n \geq 1$, on a $b_n = 0$.

(c) Calculons a_n, pour $n \geq 1$:

$$a_n = \frac{2}{T}\int_a^{a+T} f(t)\cos nt\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\mathrm{d}t = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\mathrm{d}t$$

car $t \mapsto f(t)\cos nt$ est paire ; donc :

$$a_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{2E}{\pi} t\cos nt\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi} E\cos nt\mathrm{d}t\right]$$

$$a_n = \frac{4E}{\pi^2}\int_0^{\frac{\pi}{2}} t\cos nt\mathrm{d}t + \frac{2E}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos nt\mathrm{d}t = \frac{4E}{\pi^2}J_n + \frac{2E}{\pi}I_n = \frac{2E}{\pi^2}\left(2J_n + \pi I_n\right)$$

Calculons a_{4k} :

$$a_{4k} = \frac{2E}{\pi^2}\left(2J_{4k} + \pi I_{4k}\right) = 0$$

car $J_{4k} = 0$ et $I_{4k} = 0$; en effet :

$$J_{4k} = \frac{\pi}{8k}\sin 2k\pi + \frac{1}{16k^2}\cos 2k\pi - \frac{1}{16k^2} = 0 + \frac{1}{16k^2} - \frac{1}{16k^2} = 0$$

$$I_{4k} = -\frac{1}{4k}\sin 2k\pi = 0$$

Partie C

1. Calculons a_1, a_2 et a_3 :

$$a_1 = \frac{2E}{\pi^2}(2J_1 + \pi I_1) = \frac{2E}{\pi^2}(\pi - 2 - \pi) = -\frac{4E}{\pi^2}$$

$$a_2 = \frac{2E}{\pi^2}(2J_2 + \pi I_2) = \frac{2E}{\pi^2}(-1) = -\frac{2E}{\pi^2}$$

$$a_3 = \frac{2E}{\pi^2}(2J_3 + \pi I_3) = \frac{2E}{\pi^2}\left(-\frac{\pi}{3} - \frac{2}{9} + \frac{\pi}{3}\right) = -\frac{4E}{9\pi^2}$$

2. Calculons F^2 :

$$F^2 = V_{eff}^2 = \frac{2}{T}\int_0^{\frac{T}{2}} f^2(t)\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi} f^2(t)\mathrm{d}t$$

$$F^2 = \frac{1}{\pi}\left[\frac{4E^2}{\pi^2}\int_0^{\frac{\pi}{2}} t^2\mathrm{d}t + E^2\int_{\frac{\pi}{2}}^{\pi}\mathrm{d}t\right]$$

$$F^2 = \frac{4E^2}{3\pi^3}\left[t^3\right]_0^{\frac{\pi}{2}} + \frac{E^2}{\pi}\frac{\pi}{2} = \frac{4E^2}{3\pi^3}\frac{\pi^3}{8} + \frac{E^2}{2} = \frac{E^2}{6} + \frac{E^2}{2} = \frac{2E^2}{3}$$

3. Calculons P :

$$P = a_0^2 + \frac{1}{2}(a_1^2 + a_2^2 + a_3^2) = \frac{9E^2}{16} + \frac{1}{2}\left(\frac{16E^2}{\pi^4} + \frac{4E^2}{\pi^4} + \frac{16E^2}{81\pi^4}\right) = \frac{9E^2}{16} + \frac{818E^2}{81\pi^4}$$

   Calculons $\frac{P}{F^2}$:

$$\frac{P}{F^2} = \frac{3}{2E^2}\left(\frac{9E^2}{16} + \frac{818E^2}{81\pi^4}\right) = \frac{27}{32} + \frac{409}{27\pi^4} \approx 0,999$$